## Part 1.2 Limits of functions

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## Divergence or "when the limit is infinite".

In the theory of *sequences* we say that a sequence $\{a_n\}_{n\geq 1}$ diverges if it fails to converge. There are a number of ways it could fail to converge, e.g. it could oscillate, i.e. $a_n = (-1)^n, n \geq 1$, or it could be unbounded, i.e. $a_n = n, n \geq 1$.

For certain unbounded *functions* there is a type of limit that can still be defined. The first definition below encapsulates the situation in which given a function defined on a deleted neighbourhood of $a \in \mathbb{R}$, and given any real number $K$, which we might think of as positive and large, there is some deleted neighbourhood of $a$ on which the function is *greater* than $K$. This can be repeated for each and every positive large $K$. Presumably the larger the $K$ the smaller the deleted neighbourhood. Then, if $\lim_{x\to a} f(x)$ is to be assigned a value connected in some way with the values taken by the function on deleted neighbourhoods of $a$, this value should be larger than every positive large $K$. There is **no** such possible real value! Instead we assign the symbol $+\infty$ to $\lim_{x\to a} f(x)$ .

**Definition 1.2.1** 1. *Let $f : A \to \mathbb{R}$ be a function whose domain contains a deleted neighbourhood of $a \in \mathbb{R}$. We write*

$$\lim_{x\to a} f(x) = +\infty,$$

*or say*

$$\text{“}f(x) \textbf{ tends to} +\infty \textbf{ as } x \textbf{ tends to } a\text{”}$$

*if, and only if, for all for all $K > 0$ there exists $\delta > 0$ such that if $0 < |x - a| < \delta$ then $f(x) > K$. That is:*

$$\forall K > 0, \exists \delta > 0, \forall x \in A, 0 < |x - a| < \delta \implies f(x) > K. \tag{1}$$

2. *Similarly, we write*

$$\lim_{x\to a} f(x) = -\infty,$$

*or say*

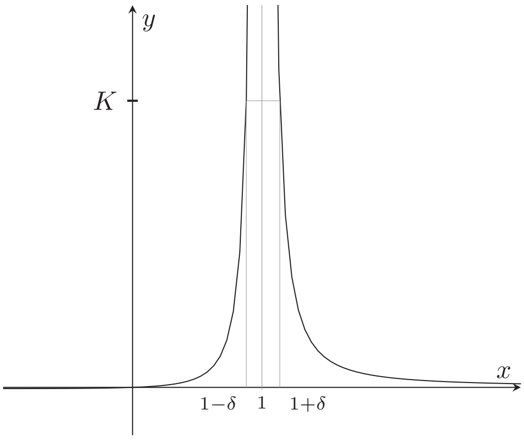
$$\text{“}f(x) \textbf{ tends to } -\infty \textbf{ as } x \textbf{ tends to } a\text{”}$$

*if, and only if, for all for all $K < 0$ there exists $\delta > 0$ such that if $0 < |x - a| < \delta$ then $f(x) < K$. That is:*

$$\forall K < 0, \exists \delta > 0, \forall x \in A, 0 < |x - a| < \delta \implies f(x) < K.$$

**Note** that here we have $f(x) < K < 0$ and so, because the numbers are negative, $|f(x)| > |K|$, i.e. $f(x)$ will be larger, in *magnitude*, than $K$!

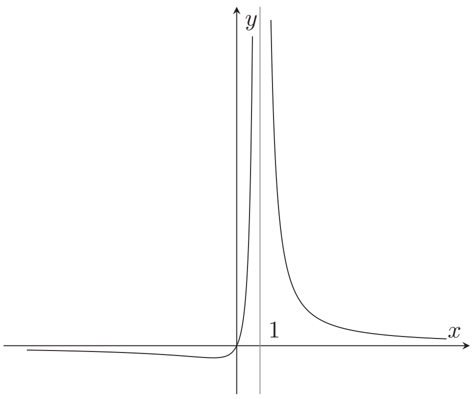
As an illustration of the $K$ - $\delta$ definition of the limit being $+\infty$ we have




**Example 1.2.2** *Verify the* $K$ - $\delta$ *definition to show that*

$$\lim_{x \to 1} \frac{x}{(x-1)^2} = +\infty.$$

Graphically, this function is very much like that used in the figure above:




**Solution** *Rough Work* Assume $0 < |x-1| < \delta$ with $\delta > 0$ to be chosen.

If we demanded that $\delta \leq 1$ then $0 < |x-1| < \delta$ would imply $0 < x < 2$, which gives

$$\frac{x}{(x-1)^2} > \frac{0}{(x-1)^2} = 0,$$

which is of no use. Thus we instead demand that $\delta$ be less than a number strictly less than 1. The 'simplest' positive number strictly less than 1 is $1/2$.

If we demand that $\delta \leq 1/2$ then $0 < |x-1| < \delta \leq 1/2$ which opens out as $-1/2 \leq x-1 \leq 1/2$, i.e. $1/2 < x < 3/2$. Thus $x$ is no smaller than $1/2$ in which case

$$\frac{x}{(x-1)^2} > \frac{1}{2(x-1)^2}.$$

Also, $0 < |x-1| < \delta$ implies $(x-1)^2 < \delta^2$ and so

$$\frac{x}{(x-1)^2} > \frac{1}{2(x-1)^2} \geq \frac{1}{2\delta^2}.$$

We demand this is $\geq K$, which rearranges as $\delta \leq 1/\sqrt{2K}$. We put these two demands on $\delta$ together as

$$\delta = \min\left(\frac{1}{2}, \frac{1}{\sqrt{2K}}\right).$$

*End of rough Work.*

**Solution** left to students ■

**Note** To show $\lim_{x\to a} f(x) = L$, with $L$ finite you have to show that $|f(x) - L| < \varepsilon$. This is normally done by finding a simpler, *upper* bound for $|f(x) - L|$ and then demanding this upper bound is $< \varepsilon$.

To show $\lim_{x\to a} f(x) = +\infty$ you have to show that $f(x) > K$. This is normally done by finding an simpler, *lower* bound for $f(x)$ and then demanding this lower bound is $> K$.

To show $\lim_{x\to a} f(x) = -\infty$ you have to show that $f(x) < K$. This is normally done by finding an simpler, *upper* bound for $f(x)$ and then demanding this upper bound is $< K$.

**Advice for the exams** It is necessary to be able to deal with inequalities concerning negative numbers. For example

$$x < y < 0 \quad \text{implies} \quad |x| > |y|;$$

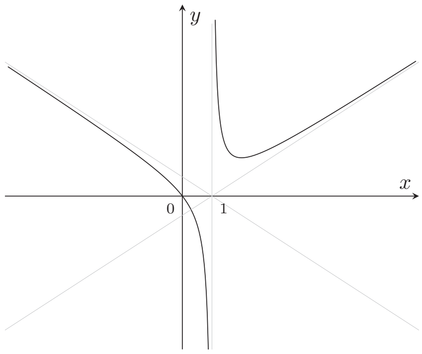
$$x < y \quad \text{implies} \quad -x > -y;$$

$$x < y \text{ and } u < 0 \quad \text{imply} \quad ux > uy.$$

And, as long as $x$ and $y$ are of the same sign, the direction of the inequality is reversed on inverting, i.e.

$$\text{if either } x > y > 0 \text{ or } 0 > x > y \quad \text{then} \quad \frac{1}{y} > \frac{1}{x}.$$

For the function illustrated in the following figure it would appear that, as $x$ tends to $+\infty$, the function $f(x)$ also tend to $+\infty$. And that as $x$ tends to 1 from above, i.e. $x \to 1+$, that $f(x)$ tends to $+\infty$.




It should not be hard for the student to supply definitions for

$$\lim_{x\to+\infty} f(x) = +\infty \text{ or } -\infty, \quad \text{and} \quad \lim_{x\to a^+} f(x) = +\infty \text{ or } -\infty.$$

Further, the student should be able to also define

$$\lim_{x\to-\infty} f(x) = +\infty \text{ or } -\infty \quad \text{and} \quad \lim_{x\to a^-} f(x) = +\infty \text{ or } -\infty.$$

In fact you will be asked to do just this in a question on the Problem Sheets.

**Note** In **none** of the cases above do we say that the limit *exists*. If we say "$\lim f(x)$ exists" we are implicitly assuming that it is finite. This is because, as in the definitions of limits at infinity, the symbols $+\infty$ and $-\infty$ are **not** real numbers. They are used simply as shorthand. To say $\lim_{x\to a} f(x) = +\infty$ is to say that $f$ satisfies (1), and there is no use of the symbol $+\infty$ in that definition.

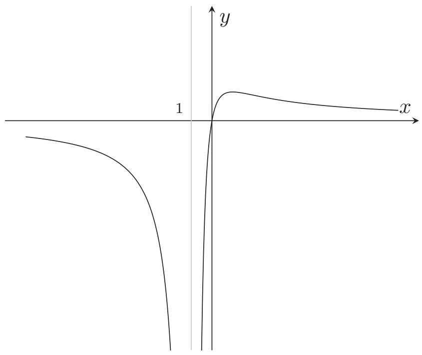
A slightly more difficult example discussed in the Tutorial.

**Example 1.2.3** *Verify the $K$ - $\delta$ definition to show that*

$$\lim_{x\to -1} \frac{x}{(x+1)^2} = -\infty.$$

**Hint** Recall that given $K < 0$ we want to find $x$ for which $f(x) < K$. We might attempt this by finding a simpler upper bound for $f(x)$ and then demanding this upper bound is $< K$ .

Graphically:




**Solution** *Rough Work* Assume $0 < |x - (-1)| < \delta$ with $\delta > 0$ to be chosen.

If $\delta \leq 1$ then $0 < |x+1| < \delta \leq 1$ would imply $-2 < x < 0$, which gives the upper bound

$$\frac{x}{(x+1)^2} < \frac{0}{(x+1)^2} = 0.$$

There is no way of demanding this is $< K < 0$. Instead demand that $\delta \leq 1/2$. Then

$$0 < |x+1| < 1/2 \implies -1/2 < x+1 < 1/2 \implies -3/2 < x < -1/2.$$

Use the upper bound on $x$ to give

$$\frac{x}{(x+1)^2} < -\frac{1}{2(x+1)^2}.$$

Then $0 < |x+1| < \delta$ implies $0 < (x+1)^2 < \delta^2$ so

$$\frac{1}{(x+1)^2} > \frac{1}{\delta^2} \quad \text{and} \quad -\frac{1}{2(x+1)^2} < -\frac{1}{2\delta^2}.$$

We now demand this is $< K$. This rearranges as

$$\delta \leq \sqrt{-\frac{1}{2K}}.$$

Remember that $K$ is negative so we are **not** taking the square root of a negative number. So $\delta = \min\left(1/2, \sqrt{-1/2K}\right)$ will suffice.

*End of Rough Work*

**Proof** Let $K < 0$ be given. Choose $\delta = \min\left(1/2, \sqrt{-1/2K}\right)$. Assume $0 < |x+1| < \delta$. Then first $|x+1| < \delta \leq 1/2$ implies $x < -1/2$.

Secondly

$$|x+1| < \delta \leq \sqrt{-\frac{1}{2K}} \implies (x+1)^2 \leq -\frac{1}{2K} \implies \frac{1}{(x+1)^2} \geq -2K. \qquad (2)$$

Combine these inequalities as

$$\frac{x}{(x+1)^2} < \left(-\frac{1}{2}\right)\left(\frac{1}{(x+1)^2}\right) \leq \left(-\frac{1}{2}\right)(-2K), \qquad (3)$$

where the direction of the inequality in (2) has changed in (3) because of being multiplied by the negative $-1/2$. Hence

$$\frac{x}{(x+1)^2} < K.$$

Thus we have verified the $K$ - $\delta$ definition of $\lim_{x\to -1} x/(x+1)^2 = -\infty$. ■

## Limit Rules

An important result says that if the limit of $f(x)$ as $x \to a$ exists and is non-zero then, for $x$ *sufficiently close* to $a$, the values of the function $f(x)$ cannot be *too large* nor *too small.*

**Lemma 1.2.4** *If* $\lim_{x\to a} f(x) = L$ *exists then there exists* $\delta > 0$ *such that*

$$\text{if } 0 < |x-a| < \delta \quad \text{then} \quad \begin{cases} \dfrac{L}{2} < f(x) < \dfrac{3L}{2} & \text{if } L > 0 \\[2ex] \dfrac{3L}{2} < f(x) < \dfrac{L}{2} & \text{if } L < 0 \\[2ex] |f(x)| < |L| + 1 & \text{for all } L. \end{cases}$$

The first two cases can be summed up as

$$\frac{|L|}{2} < |f(x)| < \frac{3|L|}{2}$$

if $L \neq 0$. A deduction from this is that if the limit at $a$ is non-zero then there exists a deleted neighbourhood of $a$ in which $f$ is non-zero.

**Proof** Assume $L > 0$. Choose $\varepsilon = L/2 > 0$ in the $\varepsilon$-$\delta$ definition of $\lim_{x\to a} f(x) = L$ to find $\delta > 0$ such that

$$\begin{aligned} 0 < |x-a| < \delta \quad &\Longrightarrow \quad |f(x) - L| < \frac{L}{2} \\ &\Longrightarrow \quad -\frac{L}{2} < f(x) - L < \frac{L}{2} \\ &\Longrightarrow \quad \frac{L}{2} < f(x) < \frac{3L}{2}. \end{aligned}$$

Assume $L < 0$. Choose $\varepsilon = -L/2 > 0$ in the $\varepsilon$-$\delta$ definition of $\lim_{x\to a} f(x) = L$ to find $\delta > 0$ such that

$$\begin{aligned} 0 < |x-a| < \delta \quad &\Longrightarrow \quad |f(x) - L| < -\frac{L}{2} \\ &\Longrightarrow \quad -\left(-\frac{L}{2}\right) < f(x) - L < -\frac{L}{2} \\ &\Longrightarrow \quad \frac{3L}{2} < f(x) < \frac{L}{2}. \end{aligned}$$

Given any $L$ choose $\varepsilon = 1$ in the $\varepsilon$ - $\delta$ definition of $\lim_{x \to a} f(x) = L$ to find $\delta > 0$ such that if $0 < |x - a| < \delta$ then $|f(x) - L| < 1$. Then, by the *triangle inequality*,

$$|f(x)| = |(f(x) - L) + L| \leq |f(x) - L| + |L| < 1 + |L|.$$

■

**Theorem 1.2.5** *Suppose that $f$ and $g$ are both defined on some deleted neighbourhood of $a$ and that both $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$ exist. Then:*

***Sum Rule**: the limit of the sum exists and equals the sum of the limits:*

$$\lim_{x \to a} [f(x) + g(x)] = L + M,$$

***Product Rule**: the limit of the product exists and equals the product of the limits:*

$$\lim_{x \to a} f(x)\, g(x) = LM,$$

***Quotient Rule**: the limit of the quotient exists and equals the quotient of the limits:*

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{L}{M}, \qquad \textit{provided that } M \neq 0.$$

**Proof** of the **Sum Rule** is left to student.

**Proof** of the **Product Rule**. Consider

$$\begin{aligned} |f(x)\, g(x) - LM| &= |f(x)\, g(x) - Lg(x) + Lg(x) - LM|, \\ &\qquad \text{"adding in zero"}, \\ &\leq |f(x)\, g(x) - Lg(x)| + |Lg(x) - LM| \\ &\qquad \text{by the triangle inequality}, \\ &= |f(x) - L|\, |g(x)| + |L|\, |g(x) - M| \qquad (4) \end{aligned}$$

Let $\varepsilon > 0$ be given. From the Lemma above $\lim_{x \to a} g(x) = M$ means there exists $\delta_1 > 0$ such that

$$0 < |x - a| < \delta_1 \implies |g(x)| < |M| + 1. \qquad (5)$$

From the definition of $\lim_{x\to a} f(x) = L$ we find that there exists $\delta_2 > 0$ such that, if $0 < |x - a| < \delta_2$ then

$$|f(x) - L| < \frac{\varepsilon}{2(|M| + 1)}. \tag{6}$$

Finally, the definition of $\lim_{x\to a} g(x) = M$ means there exists $\delta_3 > 0$ such that, if $0 < |x - a| < \delta_3$ then

$$|g(x) - M| < \frac{\varepsilon}{2(|L| + 1)}, \tag{7}$$

(where we have put a "+1" in the denominator, $2(|L| + 1)$, in case $L = 0$).

Choose $\delta = \min(\delta_1, \delta_2, \delta_3) > 0$. Assume $0 < |x - a| < \delta$. For such $x$ all the three bounds (5), (7) and (6) hold. Then, returning to (4),

$$\begin{aligned} |f(x)\,g(x) - LM| &\leq |f(x) - L|\,|g(x)| + |L|\,|g(x) - M|, \\ &< \underbrace{\frac{\varepsilon}{2(|M|+1)}}_{\text{by (6)}} \underset{\text{by (5)}}{(|M| + 1)} + |L| \underbrace{\frac{\varepsilon}{2(|L|+1)}}_{\text{by (7)}} \\ &= \left( \frac{1}{2} + \frac{1}{2} \times \underbrace{\frac{|L|}{(|L|+1)}}_{<1} \right) \varepsilon < \varepsilon. \end{aligned}$$

Thus we have verified the definition that $\lim_{x\to a} f(x)\,g(x) = LM$.

**Proof of the Quotient Rule for limits**.

Assume $\lim_{x\to a} g(x) = M$ and $M \neq 0$.

By the Lemma 1.2.4 we also have $\delta_1 > 0$ such that if $0 < |x - a| < \delta_1$ then

$$|g(x)| > \frac{|M|}{2}. \tag{8}$$

In particular $g(x) \neq 0$ and so, for such $x$ we can consider

$$\left| \frac{1}{g(x)} - \frac{1}{M} \right| = \frac{1}{|g(x)|} \frac{|g(x) - M|}{M} < \underbrace{\frac{2}{|M|}}_{\text{by (8)}} \times \frac{|g(x) - M|}{M}. \tag{9}$$

Let $\varepsilon > 0$ be given. From the definition of $\lim_{x\to a} g(x) = M$ there exists $\delta_2 > 0$ such that if $0 < |x - a| < \delta_2$ then

$$|g(x) - M| < \frac{\varepsilon\,|M|^2}{2}. \tag{10}$$

Let $\delta = \min(\delta_1, \delta_2)$ and assume $0 < |x - a| < \delta$. For such $x$ both (10) and (9) hold and we have

$$\left|\frac{1}{g(x)} - \frac{1}{M}\right| \leq \frac{2}{M^2}|g(x) - M| < \frac{2}{M^2} \times \underbrace{\left(\frac{|M|^2 \varepsilon}{2}\right)}_{\text{by (10)}} = \varepsilon.$$

Hence we have verified the definition of $\lim_{x \to a} 1/g(x) = 1/M$

I leave it to the Student to use the Product Rule to deduce

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{L}{M}$$

■

**Advice for the exam.** If asked to evaluate a limit *by verifying the* $\varepsilon - \delta$ *definition*, do **not** use the limit laws.

If asked to evaluate a limit without restriction on the method and you use a limit law **tell me** the rule being a used.